Class Functions

$\varphi : G \to GL(V)$ $\rightsquigarrow$ $\chi_\varphi : G \to \mathbb{C}$

representation   character

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a function $f : G \to \mathbb{C}$ is a **class function** if

$$f(ghg^{-1}) = f(h) \quad \forall\, g, h \in G$$

Prop: Characters are class functions

Proof:

$$\chi_\varphi(ghg^{-1}) = Tr(\varphi_{ghg^{-1}})$$

$$= Tr(\varphi_g \circ \varphi_h \circ (\varphi_g)^{-1})$$

$$= Tr(\varphi_h) = \chi_\varphi(h)$$

Write: $L(G) := \{ \text{functions } f : G \to \mathbb{C} \}$

$$\cup\!|$$

$L^c(G) := \{ \text{class function} \}$

Note: both $\mathbb{C}$-vector spaces

$\dim L(G) = |G|$

$\dim L^c(G) =$ number of conjugacy classes in G

$L(G) \sim$ "inner product space"

Define: $f_1, f_2 \in L(G)$

$$\langle f_1, f_2 \rangle := \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)} \in \mathbb{C}$$

$$\langle -, - \rangle : L(G) \times L(G) \longrightarrow \mathbb{C}$$

$$\langle f_1, f_2 \rangle := \frac{1}{|G|} \sum_{g \in G}' f_1(g) \overline{f_2(g)}$$

<u>Properties</u>:

(1) $\langle \lambda_1 f_1 + \lambda_2 f_2 , f \rangle = \lambda_1 \langle f_1, f \rangle + \lambda_2 \langle f_2, f \rangle$

$f_1, f_2, f \in L(G)$, $\lambda_1, \lambda_2 \in \mathbb{C}$

(2) $\langle f_2, f_1 \rangle = \overline{\langle f_1, f_2 \rangle}$

(3) $\langle f, f \rangle \in \mathbb{R}_{>0}$ if $f \in L(G)$, $f \neq 0$

i.e. $\langle f_1, f_2 \rangle$ is a <u>Hermitian inner product</u> on the $\mathbb{C}$-vector space $L(G)$

(1),(2): $\langle f, \lambda_1 f_1 + \lambda_2 f_2 \rangle = \bar{\lambda}_1 \langle f, f_1 \rangle + \bar{\lambda}_2 \langle f, f_2 \rangle$

<u>Orthonormal subset</u> of $L(G)$ or $L^c(G)$:

$f_1, \ldots, f_s$ such that

$$\langle f_i, f_j \rangle = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

<u>orthonormal basis</u>: o.n. subset $f_1, \dots, f_s$ which is also a basis (of $L(G)$, or $L^c(G)$)

<u>o.n.b</u>. $f_1, \dots, f_s$ of $V \leq L(G)$, then

$$f = \sum_{k=1}^{s} \langle f, f_k \rangle f_k \qquad \text{if } f \in V$$